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Chern-Simons solitons, chiral model, and (affine) Toda model on noncommutative space

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Abstract: We consider the Dunne-Jackiw-Pi-Trugenberger model of a U(N) Chern-Simons gauge theory coupled to a nonrelativistic complex adjoint matter on noncommutative space. Soliton configurations of this model are related the solutions of the chiral model on noncommutative plane. A generalized Uhlenbeck's uniton method for the chiral model on noncommutative space provides explicit Chern-Simons solitons. Fundamental solitons in the U(1) gauge theory are shaped as rings of charge $\mathfrak{n}$ and spin $-\mathfrak{n}$ where the Chern-Simons level $\mathfrak{n}$ should be an integer upon quantization. Toda and Liouville models are generalized to noncommutative plane and the solutions are provided by the uniton method. We also define affine Toda and sine-Gordon models on noncommutative plane. Finally the first order moduli space dynamics of Chern-Simons solitons is shown to be trivial.

Keywords: Field Theories in Lower Dimensions, Chern-Simons Theories, Non-Commutative Geometry, Integrable Equations in Physics.

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Contents

1. Introduction and conclusion

There has been considerable interest in finding solitons on noncommutative space. (For reviews, see refs. [1, 2, 3] for example.) Classically well-known solitons, like instantons, magnetic monopoles, vortices on commutative space have found also on the theories defined on noncommutative plane. In addition, there are certain solitons whose existence is possible only on noncommutative plane. Some are intrinsically defined only on noncommutative plane [4]. Others are extension of solitons to the U(1) gauge theories [5].

Sometime ago Dunne, Jackiw, Pi and Trugenberger (DJPT) have studied solitons of zero energy in a Chern-Simons gauge theory coupled to nonrelativistic complex matter field in adjoint representation [6, 7]. Once a consistent ansatz is made for the field configuration, the self-dual equations for these Chern-Simons solitons can be reduced to the equations for the Liouville and Toda models whose solutions are known [6]. In addition the equations for the affine Toda and sinh-Gordon models were shown to be derivable from a consistent ansatz. In ref. [7] all Chern-Simons soliton configurations have been found to be related to the solutions of the chiral theory, which can be found by Uhlenbeck's uniton method [8, 9]. In addition the solutions for the Toda model were recovered by the uniton method.

In this work we investigate the extension of the DJPT model to noncommutative space, which is possible only for $U(N)$ gauge group. We find that the correspondence between the solutions of the self-dual equation and the chiral model works equally well on noncommutative plane. The uniton method works as well with some caveat. Interestingly the Chern-Simons solitons on U(1) gauge theory are made of fundamental 'ring' shape configurations of charge $\mathfrak{n}$ and spin $-\mathfrak{n}$ with the Chern-Simons level $\mathfrak{n}$. A consistent ansatz for (affine) Toda model can be found also on noncommutative space, even though the first order equations cannot be reduced to second order equations as on commutative space. Our equations can be regarded as a generalization of Liouville, sinh-Gordon, and (affine) Toda models to noncommutative space. Using the uniton method, we find also the solution of the generalized Liouville and Toda equations on noncommutative plane. Note that the explicit solutions of Toda models on commutative space cannot be translated easily to those on noncommutative space. In addition, we show that the moduli space dynamics of Chern-Simons solitons is trivial.

The Chern-Simons solitons of Dunne-Jackiw-Pi-Trugenberger [6, 7] belong to a large class of self-dual solitons appearing in Chern-Simons theories coupled to matter fields, where the global or local U(1) symmetry is essential for their existence. (See for reviews [10].) They could come in varieties as vortices in asymmetric phase, q-balls or q-balls with vortices in symmetric phases. The DJPT solitons can be regarded as the nonrelativistic limit of q-balls with vortices in middle of certain class of the relativistic theories. What is interesting about DJPT model is that the lowest or zero energy solutions satisfy the self-dual equations which is dimensional reduction of the self-dual Yang-Mills equation on space with (2,2) signature. Indeed they made a correspondence between the self-dual equations with the equations for the chiral model [6], and in ref. [7] the complete solutions are found by Uhlenbeck's uniton method [8, 9].

In addition, they found that upon choosing a consistent ansatz, the self-dual equations are reduced to Toda or affine Toda equations. For that of SU(2), these equations can be shown to contain Liouville and sinh-Gordon equations, respectively. In DJPT papers, they have provided a version of the general solution of SU(N) Toda equation which can be also obtained from the uniton method. Their work can be also understood in the context of Ward's conjecture [11], which associates all integrable models to the dimensional reduction of self-dual Yang-Mills equations on flat four dimensional space with $(4,0)$ or $(2,2)$ signature.

The solitonic physics on noncommutative space has been considered recently. New features of solitons can appear on noncommutative space. Some solitons which can collapse to points and disappear on ordinary space reaches a finite size on noncommutative space. On noncommutative space their moduli space becomes complete in short distance. Instantons in the U(N) gauge theory on noncommutative four space provides a prototype of noncommutative example [5].

A pure neutral scalar field theory could have nontrivial solitons on noncommutative space [4]. In large noncommutative limit, a projection operator on Hilbert space defined by the noncommutative spatial coordinates becomes a solitonic configuration. The dynamics of these solitons are explored in ref. [12, 13]. These operators will play a role in the physics of Chern-Simons solitons. The chiral models on noncommutative space has been explored by the dressing methods [14], where some of solutions are explored in detail. Our approach here by unitons is somewhat different from their approach and probably provides all soliton solutions in one setting. Our work on unitons on noncommutative space are more closely related to the self-dual solutions on CP_N and grassmannian models, which has been explored in many directions [15, 16, 17].

Our generalization of the DJPT model to noncommutative space is straightforward. The Chern-Simons solitons appear as a lump on noncommutative plane with integer global U(N) charge and integer spin, whose minimum value is $\mathfrak{n}$ and $-\mathfrak{n}$ respectively with the Chern-Simons level $\mathfrak{n}$, which should be integer upon quantization [18, 19, 20]. The general

Chern-Simons solitons are composite of finite number of fundamental solitons of minimum charge. When the gauge group is U(1), each fundamental soliton has two moduli parameters accounting for its position. When the gauge group is larger, there are also scale and internal orientation parameter, exactly like instantons on noncommutative four dimensional space.

The configurations for Chern-Simon solitons on noncommutative space cannot be found easily by trying to translate the explicit solutions of Toda equations on commutative space. Here we relate the self-dual equations for Chern-Simon solitons on noncommutative space to the equation of U(N) chiral model on noncommutative space. Fortunately the chiral equation can be solved by the uniton method. On commutative space the unitons provide the most general solution. While it is not yet proven, this seems to be case also on noncommutative space. Thus the uniton method seem to provide general Chern-Simons soliton configurations.

In addition, the reduction to (affine) Toda model by a consistent ansatz works equally well on noncommutative space once the gauge group is U(N) and the ordering of operators are carefully taken care of. This leads to a generalization of (affine) Toda model to noncommutative space. For U(2) case, the self-dual equation with a proper consistent ansatz leads to Liouville and sinh-Gordon equations. In our generalization, we still have coupled first order equations on noncommutative space, which cannot be reduced to coupled second order equations in general. In recent works on integrable models on noncommutative space [21], the coupled second order equations are generalized to noncommutative plane. It remains to be seen how two generalizations to noncommutative space are related each other. While our work is focused on euclidean noncommutative plane, it is straightforward to reduce the self-dual Yang-Mills equations with (2,2) signature along the Minkowski noncommutative plane. Again the equations obtained from this angle would be coupled first order equations.

Finally we study the moduli space dynamics of Chern-Simons solitons briefly, which is first order in time-derivative. There are moduli parameters for Chern-Simons solitons. The first order lagrangian would lead to a first order lagrangian on moduli space. It turns out to be trivial as the first order induced kinetic term on moduli space vanishes. This is consistent with the fact that the total angular momentum of many Chern-Simons solitons is independent of their position and so no nontrivial interaction required for spin-statistics theorem is expected. There are several works on moduli space dynamics of solitons in CP_N and chiral models on noncommutative plane, which as some relevance to our work here [12, 13, 14, 16, 22]. They are basically looking at the second order kinetic term, which leads to the smooth moduli space metric of identical particles.

In addition, it would be interesting to find out nontrivial time-dependent solutions of Chern-Simons solitons which is not captured in the moduli space dynamics. The U(1) Chern-Simons theory coupled to a nonrelativistic matter field in fundamental representation has been studied [23] and some time-dependent solutions has been found [24]. A pursuit along this direction may be profitable.

The plan of this work is as follows. In section 2, we introduce the DJPT model on two dimensional noncommutative space, and explore some basic properties. In section 3, we relate the self-dual equations for Chern-Simons solitons to the equation of the chiral model.

We review and extend the uniton method to noncommutative space. We also study some basic properties of Chern-Simons soliton solutions. In section 4, we introduce a consistent ansatz and obtain (affine) Toda model on noncommutative space. We also study a explicit solution obtained by the uniton method. Finally in section 5, we show that the first order moduli space dynamics of the Chern-Simons solitons is trivial.

2. DJPT theory on noncommutative space

Our space is a two dimensional noncommutative space with noncommuting coordinates $(x, y) = (x^1, x^2)$ satisfying

$$[x, y] = i\theta\,, \tag{2.1}$$

where $\theta > 0$ without lose of generality. The complex coordinate variables are $z = (x+iy)/2$ and $\bar{z} = (x-y)/2$. The creation and annihilation operators are defined as

$$a = \frac{x+iy}{\sqrt{2\theta}} = \sqrt{\frac{2}{\theta}}\, z\,, \qquad \bar{a} = \frac{x-iy}{\sqrt{2\theta}} = \sqrt{\frac{2}{\theta}}\, \bar{z}\,, \tag{2.2}$$

which satisfy $[a, \bar{a}] = 1$. The Hilbert space of harmonic oscillator is given by $\{|n\rangle, n = 0, 1, 2, \ldots\}$ such that $a|0\rangle = 0$ and $|n\rangle = \bar{a}^n/\sqrt{n!}|0\rangle$. The three dimensional space-time coordinates are $(x^0, x^1, x^2) = (t, x, y)$. Any field $\phi(x, y)$ on noncommutative space becomes an operator on the Hilbert space. The space integration becomes the trace over this Hilbert space $\int d^2x = 2\pi\theta\ \mathrm{Tr}$. The spatial derivatives $\partial_i\phi$ can be represented by operators $\partial_x\phi = i[y, \phi]/\theta$, $\partial_y\phi = -i[x, \phi]/\theta$. In the complex variables

$$\partial_+ = \partial_x + i\partial_y = \partial_{\bar{z}} = \frac{2}{\theta}[z, \,]\,, \qquad \partial_- = \partial_x - i\partial_y = \partial_z = \frac{2}{\theta}[\bar{z}, \,]\,. \tag{2.3}$$

The antihermitian gauge fields $A_\mu = A_t, A_x, A_y$ for the U(N) gauge group define a covariant derivative on the scalar field ϕ in the adjoint representation via $D_i\phi = \partial_i\phi + [A_i, \phi]$, where ϕ and A_i are N by N matrices. On our noncommutative plane, the covariant derivatives can be rewritten as

$$D_x\phi = \frac{i}{\theta}[Y, \phi]\,, \qquad D_y\phi = \frac{-i}{\theta}[X, \phi]\,, \tag{2.4}$$

where the covariant position operators are

$$X = x + i\theta A_y\,, \qquad Y = y - i\theta A_x\,. \tag{2.5}$$

The covariant complex position operator and its conjugate are $Z = (X + iY)/2$ and $\bar{Z} = (X - iY)/2$. The complex covariant derivatives are

$$D_+ = \partial_+ + A_+ = \frac{2}{\theta}[Z, \,]\,, \qquad D_- = \partial_- + A_- = \frac{2}{\theta}[\bar{Z}, \,]\,, \tag{2.6}$$

where $A_\pm = A_x \pm iA_y$. In complex coordinates, the field strength becomes

$$F_{+-} = \partial_+ A_- - \partial_- A_+ + [A_+, A_-] = -2iF_{xy}\,. \tag{2.7}$$

The theory we consider here is the U(N) Chern-Simons gauge theory coupled to a nonrelativistic complex bosonic field ϕ in the adjoint representation, which is a simple generalization of the SU(N) DJPT model. (A pair of hermitan fields in the adjoint representation are combined to a complex field in the adjoint presentation.) The gauge field A_μ is N by N antihermitian matrices and the matter field ϕ is N by N complex matrix. Its lagrangian is

$$L = -\frac{\mathfrak{n}}{4\pi}\int d^2x \left\{\epsilon^{\mu\nu\rho}\,\mathrm{tr}\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) - i\,\mathrm{tr}\left(\phi^\dagger D_0\phi\right)\right\} - H\,, \tag{2.8}$$

where $D_\mu\phi = \partial_\mu\phi + [A_\mu, \phi]$ and tr is a trace over $N\times N$ matrices.

Up on the quantization of the Chern-Simons theory on noncommutative plane, the Chern-Simons level $\mathfrak{n}$ should be quantized in integer for consistency [18, 19, 20] for all naturall number N. Our choice of the scalar field normalization is for convenience. In order to obtain the standard kinetic energy for the matter field, we need to change the normalization of the ϕ field, with possible charge conjugation. The conserved energy is

$$H = \frac{\pi|\mathfrak{n}|}{m}\int d^2x\,\mathrm{tr}\left(2D_i\phi^\dagger D_i\phi - [\phi,\phi^\dagger]^2\right). \tag{2.9}$$

The mass parameter m is positive. The potential is negative and so attractive. With the standard normalization of the ϕ field m is the kinetic mass and the attractive interaction is inversely proportional to $\mathfrak{n}$.

On noncommutative plane the allowed gauge group is U(N) instead of SU(N). Under the U(N) gauge transformation,

$$\phi \to U^\dagger(x)\phi U(x)\,, \qquad A_\mu \to U^\dagger(x)A_\mu U(x) + U(x)\partial_\mu U(x)\,. \tag{2.10}$$

The Gauss's law constraint is

$$F_{+-} = -2iF_{xy} = [\phi,\phi^\dagger]\,. \tag{2.11}$$

There is a global U(1) symmetry $\phi\to e^{i\alpha}\phi$, whose conserved charge is

$$Q = \frac{\mathfrak{n}}{4\pi}\int d^2x\,\mathrm{tr}\,\phi\phi^\dagger\,, \tag{2.12}$$

which is expected to be quantized in integer.

To consider the lowest energy configuration for a given charge, we first note that

$$\mathrm{tr}(D_i\phi^\dagger D_i\phi) = \mathrm{tr}\Big((D_+\phi^\dagger)(D_-\phi) - i[\phi,\phi^\dagger]F_{12}\Big) + i\epsilon_{ij}\partial_i\,\mathrm{tr}(\phi^\dagger D_j\phi)\,. \tag{2.13}$$

With the Gauss law (2.11), the conserved energy (2.9) for any localized configuration becomes

$$H = \frac{2\pi|\mathfrak{n}|}{m}\int d^2x\,\mathrm{tr}(D_+\phi^\dagger)(D_-\phi)\,, \tag{2.14}$$

which is nonnegative. Its minimum, zero energy, can be achieved by localized configurations satisfying the Gauss law constraint (2.11) and the self-dual equation

$$D_-\phi = 0\,. \tag{2.15}$$

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The physics of these zero energy configurations, Chern-Simons solitons, on noncommutative plane is the main concern here. Note that the Gauss law (2.11) and selfdual equation (2.15) can be regarded as a dimensional reduction of the selfdual Yang-Mills equation on four dimensional space of signature (2,2).

Under the infinitesimal rotation $\delta x^i = \epsilon_{ij} x^j$ with the corresponding field transformation

$$\delta\phi = \frac{1}{2}\epsilon_{ij}(X^i(D_j\phi) + (D_j\phi)X^i)\,, \tag{2.16}$$

$$\delta A_i = -\frac{1}{2}(X^i F_{12} + F_{12}X^i)\,, \tag{2.17}$$

leaves the action invariant modulo a gauge transformation. Note that the symmetrized transformation of the field is essential on noncommutative plane. The corresponding conserved angular momentum [19] is

$$J = \frac{\mathfrak{n} i}{16\pi}\int d^2x\;\epsilon_{ij}\;\mathrm{tr}\;X^i\Big((D_j\phi\phi^\dagger - \phi D_j\phi^\dagger) + (\phi^\dagger D_j\phi - D_j\phi^\dagger\phi)\Big)\,, \tag{2.18}$$

which has a gauge-covariant density. The rotational symmetry group is U(1) on the noncommutative plane. The conserved linear momentum can be obtained by noting that under shifting $X^i \to X^i + a^i$, $J \to J + \epsilon_{ij}a^i P^j$. For the self-dual configurations satisfying (2.11) and (2.15), the above angular momentum becomes

$$J = \frac{\mathfrak{n}}{16\pi}\int d^2x \Big\{ Z(\phi D_-\phi^\dagger + D_-\phi^\dagger\phi) + \bar{Z}(D_+\phi\phi^\dagger + \phi^\dagger D_+\phi)\Big\}\,. \tag{2.19}$$

Integration by part leads to

$$J = -Q\,. \tag{2.20}$$

The charge and angular momentum quantizations are consistent to one another.

3. Chiral model and unitons

Similar to the commutative case, one can easily map the Gauss law and the self-dual equation to the field equation for the chiral model of $\mathrm{U}(N)$ group on noncommutative plane. As in ref. [6], we introduce a new 'auxiliary gauge field'

$$\mathcal{A}_+ = A_+ - \phi\,, \qquad \mathcal{A}_- = A_- + \phi^\dagger\,, \tag{3.1}$$

which has zero field strength

$$\mathcal{F}_{+-} = \partial_+\mathcal{A}_- - \partial_-\mathcal{A}_+ + [\mathcal{A}_+, \mathcal{A}_-] = 0 \tag{3.2}$$

due to the Gauss law (2.11) and the self-dual equation (2.15). (One can introduce arbitrary phase difference between A_+ and ϕ of $\mathcal{A}_+$ field, which corresponds to the spatial U(1) and global U(1) rotations, and so immatiarial.) Thus the gauge field $\mathcal{A}_\pm$ should be pure gauge which we choose to be zero. (Later we will choose a gauge transformation of this $A_+ = \phi$ configuration, leading to nonzero $\mathcal{A}_+$.) Thus,

$$A_+ = \phi\,, \qquad A_- = -\phi^\dagger\,, \tag{3.3}$$

which solves the Gauss law. Now the self-dual equation (2.15) and its hermitan conjugate become

$$\partial_- \phi = [\phi^\dagger, \phi]\,, \qquad \partial_+ \phi^\dagger = [\phi^\dagger, \phi]\,, \tag{3.4}$$

which are equivalent to two equations

$$\partial_+ \phi^\dagger - \partial_- \phi = 0\,, \tag{3.5}$$

$$\partial_+ \phi^\dagger + \partial_- \phi + 2[\phi, \phi^\dagger] = 0\,. \tag{3.6}$$

Introducing another auxiliary gauge field $C_+ = 2\phi$ and $C_- = -2\phi^\dagger$, the second equation (3.6) becomes the zero field strength equation for $C_\pm$. This simplies that $C_\pm$ are pure gauge or

$$\phi = \frac{1}{2} h^{-1} \partial_+ h\,, \qquad \phi^\dagger = -\frac{1}{2} h^{-1} \partial_- h\,, \tag{3.7}$$

for a map $h(x)$ from the noncommutative plane to U(N) group. The first equation (3.5) implies that h satisfies

$$\partial_+ (h^{-1} \partial_- h) + \partial_- (h^{-1} \partial_+ h) = 0\,, \tag{3.8}$$

which is the equation for the U(N) chiral model on noncommutative plane.

This chiral equation can be derived from the stationary point of the nonnegative energy functional

$$\mathcal{H}_{\rm chiral} = -\frac{1}{2} \int d^2 x\, {\rm tr} (h^{-1} \partial_i h)^2\,. \tag{3.9}$$

As the chiral field h is related to the complex scalar field ϕ by the relation (3.7), the conserved U(1) charge (2.12) of the Chern-Simons theory becomes

$$Q = -J = \frac{\mathfrak{n}}{8\pi} \mathcal{H}_{\rm chiral}\,. \tag{3.10}$$

On the commutative plane there exists an algorithm to find all solutions of the chiral model which have the finite energy $\mathcal{H}_{\rm chiral}$ by using the so-called unitons [8, 9]. One key aspect of the theorem is a method to generate new solutions from old ones, by the so-called 'the addition of a uniton' procedure. It works equally well on noncommutative plane as we will see now. For a given solution h_0 of the chiral model equation, one defines a 'uniton' factor with respect to h_0 to be $2p-1$ with a hermitian projection operator p which satisfies two equations,

$$(1-p)(\partial_+ p + A^0_+ p) = 0,\ \ p A^0_+ (1-p) = 0 \tag{3.11}$$

where $A^0_\pm = \frac{1}{2} h_0^{-1} \partial_\pm h_0$. Then the operator $h = h_0(1-p)$ is also a solution of the chiral model equation because for such h_0 and p,

$$h^{-1} \partial_+ h = 2(\partial_+ p + A^0_+)\,, \qquad h^{-1} \partial_- h = 2(-\partial_- p + A^0_-)\,. \tag{3.12}$$

Let us now put forward a conjecture on noncommutative plane which is a simple generaliaztion of the theorem due to K.Uhlenbeck [8] and refined by J.C. Wood [9]. We generalize the version appeared in ref. [7] to the U(N) chiral model on noncommutative plane, where the operator ordering is crucial matter.

Conjecture 1. *Every solution h of the* U(N) *chiral model equation (3.8) with finite energy (3.9) on noncommutative plane can be uniquely factorized as a product*

$$h = (2p_1 - 1)(2p_2 - 1) \cdots (2p_m - 1) \tag{3.13}$$

with $m \leq N - 1$ where (a) each p_i is a hermitian $N \times N$ projection operator on noncommutative plane, and (b) with a definition $h_j = (2p_1 - 1)(2p_2 - 1) \cdots (2p_j - 1)$, $2p_{j+1} + 1$ is a uniton with respect to h_j for all $j = 1, \ldots, m - 1$.

We do not know whether above conjecture is true or not. It remains to be explored further. There exists also a theorem by G. Valli [25] on the value of the energy (3.9) for a solution h of the chiral model on commutative plane. Its generalization to noncommutative space becomes as follows:

Conjecture 2. *Let h be a solution of the chiral model equation (3.8) on noncommutative plane, which can be expressed as in eq. (3.13). Then the conserved energy (3.9) takes a nonnegative integer mulitiple of* 8π.

This theorem holds for all configurations we will study later and is expected to hold on noncommutative space. Thus the charge and spin of Chern-Simons solitons would be positive integer multiples of $\mathfrak{n}$ and $-\mathfrak{n}$. We could say that a general Chern-Simons solitons are made of integer number of fundamental Chern-Simons solitons of charge $\mathfrak{n}$ and spin $-\mathfrak{n}$.

To quadratic order, elementary particles describes by the field ϕ carries unit charge and so one could regard the fundamental Chern-Simons solitons as a composite of $\mathfrak{n}$ elementary particles bound together by an attractive force of strength $1/\mathfrak{n}$. While we expect elementary particles are anyons with nontrivial spin-statistics, even may with nonabelian braid group, fundamental Chern-Simons solitons carry integer spin upon quantization as $\mathfrak{n}$ is quantized in integer.

The weak coupling limit is the large $\mathfrak{n}$ limit, and one can see that the quartic attractive interaction becomes weak as it is proportional to $1/\mathfrak{n}$ after a proper normalization of the scalar field. Thus our classical picture of Chern-Simons solitons becomes better for large $\mathfrak{n}$. While it is in principle possible for our Chern-Simons solitons to have purely nonabelian statistics, we will see later that no such interaction is apparent on first order dynamics on the space of moduli parameter.

There are studies of some explicit chiral solutions by using unitons [26, 27] on commutative plane, which would have also application on noncommutative space. More recently, the solutions of the chiral model on noncommutative plane have been explored by the dressing method [14]. Here our goal is to exploring the role of unitons on both Chern-Simons solitons and the solutions of chiral model on noncommutative plane. The understanding obtained from the soliton solutions on CP_N model plays a key role here [15, 17].

Assuming above two conjectures are true, let us explore their imiplications on Chern-Simons solitons. First let us consider the simplest case, the the chiral and DJPT model of U(1) group on noncommutative plane. Note that these models with U(1) group do not have nontrivial solutions on commutative plane. The uniton solution would be

$$h = (2p - 1) \tag{3.14}$$

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such that

$$(1-p)\partial_+ p = 0 \tag{3.15}$$

which is equivalent to $(1-p)ap = 0$. The hermitan projection operator satisfying $p^2 = p$ has been extensively studied in the study of solitons on noncommutative plane [4]. The projection operator for K number of solitons at positions $z_r = (x_r + iy_r)/2$ is given by

$$P_K = \sum_{r,s=1}^{K} |z_r\rangle\langle z_r|z_s\rangle^{-1}\langle z_s| \tag{3.16}$$

with

$$|z_r\rangle = \exp\left(-\frac{|z_r|^2}{\theta} + \sqrt{\frac{2}{\theta}}\, z_r a^\dagger\right)|0\rangle \tag{3.17}$$

Note that $(z - z_r)|z_r\rangle = 0$ for all r. Interestingly, the above projection operator P_K satisfies the equation (3.15), implying $h = h^{-1} = 2P_K - 1$ is the solution of the chiral equation. For p satisfying (3.15) and having finite trace, the conserved hamiltonian (3.9) becomes

$$\mathcal{H}_{\text{chiral}} = -8\pi\,\mathrm{Tr}[a,p][\bar{a},p] = -8\pi\,\mathrm{Tr}(ap\bar{a} - pa\bar{a}) = 8\pi\,\mathrm{Tr}\,p\,. \tag{3.18}$$

Thus for $p = P_K$, the above quantity would be $8\pi K$, confirming Conjecture II.

For such a solution of the chiral equation, the soliton solution of the DJPT model would be given by $A_+ = \phi = \partial_+ P_K$. The total charge and angular momentum are $Q = -J = \mathfrak{n}K$. When particles are all coming together the projection operator becomes $P_K = \sum_{r=0}^{K-1}|r\rangle\langle r|$, and so the magnetic field of the corresponding Chern-Simons soliton becomes

$$F_{+-} = [\phi,\phi^\dagger] = K\big(|K-1\rangle\langle K-1| - |K\rangle\langle K|\big) \tag{3.19}$$

so that the net magnetic flux vanishes. Our Chern-Simons solitons can be regarded as a nonrelativistic limit of Q-balls with vortex at center in the symmetric phase of the relativistic Chern-Simons Higgs theory [28, 29, 30]. The above field strength shows the remnant of this configuration. On the right side the first term is the vortex contribution and the second term is the Q-ball contribution. The angular momentum could be regarded as the difference between the vortex contribution and the Q-ball contribution. Our solutions describe a composite of K number of fundamental Chern-Simons solitons of ring shape.

In the U(N) theory, we consider a N by $N' < N$ matrix

$$M = \begin{pmatrix} f_{11}(z) & f_{12}(z) & \cdots & f_{1N'}(z) \\ \cdot & \cdot & \cdots & \cdot \\ f_{N1}(z) & f_{N2}(z) & \cdots & f_{NN'}(z) \end{pmatrix} \tag{3.20}$$

where $f_{ij}(z)$ are polynomials of z. With a projection operator defined as

$$p = M\frac{1}{\bar{M}M}\bar{M}\,. \tag{3.21}$$

It satisfies $(1-p)\partial_+ p = 0 = (\partial_+ p)p$, implying $h = 2p-1$ satisfies the chiral equation. Thus the above projection operator p can be regarded as a single uniton. The N by N' matirx operator

$$\Phi = M(\bar{M}M)^{-1/2} \tag{3.22}$$

satisfies $\Phi\Phi^\dagger = p$ and $\Phi^\dagger\Phi = 1_{N'N'}$. It can be regarded as an harmonic map from the noncommutative plane to an Grassmann manifold [15, 16, 17].

This contrasts to the commutative case where f_{ij} can be rational analytic functions of z. As we have seen that a single uniton solution is closely related to the CP_{N-1} or Grassmann solitons on noncommutative plane, in which case the entries of M being polynomial, not a rational function, is essential for consistency [15]. The point is that a rational function has poles where it becomes not holomorphic due to Dirac-delta function. It can be neglected on commutative space but not on noncommutative space, and so the correspodning projection operator p is not a uniton any more, and leads to un-quantized topological charge.

The trace of the above p is infinite. With $h = 2p-1$, the chiral energy becomes the topological charge for Φ as

$$\begin{aligned} \mathcal{H}_{\text{chiral}} &= 4\pi\theta \ \operatorname{Tr}\operatorname{tr}_N \partial_+ p(1-p)\partial_- p \\ &= 4\pi\theta \operatorname{Tr}\operatorname{tr}_{N'} \partial_+ \bar{\Phi}(1-p)\partial_- \Phi \\ &= 4\pi\theta \operatorname{Tr}\operatorname{tr}_{N'} \partial_+ \left(\frac{1}{\bar{M}M}\bar{M}\partial_- M\right). \end{aligned} \tag{3.23}$$

The argument in ref. [15] shows that the above quantity becomes 8π times the highest degree of the polynomials appear in M.

Especially with U(2) group, we consider a single uniton solution defined by a 2 dimensional vector

$$M = \begin{pmatrix} f(z) \\ c \end{pmatrix} \tag{3.24}$$

where $f(z)$ is the K-th order polynormial

$$f(z) = \prod_{r=1}^{K}(z - z_r). \tag{3.25}$$

The corresponding projection operator becomes

$$p = \begin{pmatrix} f(z) \\ c \end{pmatrix} (\bar{f}f + |c|^2)^{-1}(\bar{f}(z), \bar{c}). \tag{3.26}$$

In the limit $c \to 0$, this projection operator approaches

$$p \to \begin{pmatrix} 1 & 0 \\ 0 & P_K \end{pmatrix} \tag{3.27}$$

with P_K in eq. (3.16). The reason is that P_K is only projection operator such that $(z - z_r)P_K = 0$ for all r. The above arguement shows that the chiral energy can be shown to be $8\pi K$. For $K=1$, $f(z)=z-z_1$ and so the chiral energy is 8π, and the charge and spin of the Chern-Simons soliton is $Q=-J=\mathfrak{n}$, indicating that it is a fundamental one.

The counting of the parameters of fundamental soliton is a bit subtle. On chiral model and CP_N model, there are vacuum parameters, which need to be specified. Those parameters has infinite inertia and so does not change with time. The solutions of the chiral and CP_N models would then have additional parameters. However, in our DJPT model, there is no nontrivial vacuum parameters as our scalar field ϕ vanishes at infinity. For a single fundamental soliton on U(N) theory would have not only position moduli parameters, but also scaling and internal orientation paramters. When there are several solitons presents, there would be relative size and orientation paramters also. It would be interesting to find out the right counting.

4. Toda-model on noncommutative plane

On commutative plane, a class of Chern-Simons soliton solution has been associated with solutions of the SU(N) Toda-model [6, 7]. Especially for $N=2$, the Toda-model is equivalent to the Liouville model. On noncommutative plane, the integrable generalization of U(N) Toda-equation has not been stated as far as we know, even though there are several works on the generalization of integrable models on noncommutative plane. We will show here that a straightforward extension of the work in ref. [6, 7], leads to a generalization of integrable Liouville and Toda-models on noncommutative and a large class of its solutions. By examiming some simple example, we will see this generalization is still highly nontrivial.

As on commutative plane, we start with an consistent ansatz for the field configuration,

$$A_+ = \mathrm{diag}(E_1, E_2, \ldots, E_N)\,, \qquad (\phi)_{ab} = \delta_{a,b-1} h_a \quad (a=1,2,\ldots,N-1) \tag{4.1}$$

with N components E_a and $N-1$ components h_a. The self-dual equations (2.11) and (2.15) on noncommutative plane for this ansatz become

$$\partial_+ h_a + E_a h_a - h_a E_{a+1} = 0 \qquad (a=1,2,\ldots,N-1) \tag{4.2}$$

$$\begin{aligned}
\partial_+(-E_1^\dagger) - \partial_- E_1 + [(-E_1^\dagger), E_1] &= h_1 h_1^\dagger \\
\partial_+(-E_a^\dagger) - \partial_- E_a + [(-E_a^\dagger), E_a] &= -h_{a-1}^\dagger h_{a-1} + h_a h_a^\dagger \qquad (a=2,\ldots,N-1) \\
\partial_+(-E_N^\dagger) - \partial_- E_N + [(-E_N^\dagger), E_N] &= -h_{N-1}^\dagger h_{N-1}\,.
\end{aligned} \tag{4.3}$$

On noncommutative space the gauge field A_+ is not traceless and so one could not solve the first set of equations (4.2) for E_a to reduce the above coupled linear equations to the second order equations for h_a only. One has to live with these coupled first order equations (4.2) and (4.3) for E_a and h_a and regard them to define the U(N) Toda model on noncommutative plane. For $N=2$, the above equations become the generalization of the Liouville equation on noncommutative plane. On commutative such reduction can be done and one ends up with the second order equations for $\rho_a = |h_a|^2$ which define the Toda model.

To find the solution of the above equations on noncommutative plane, let us start with a set of explicit solutions, the so-called 'Toda-type', in terms of unitons on commutative plan [7]. we will show that its main gist works out on noncommutative plane as well with due care on ordering of operators. To find for the solutions in U(N) theory, we start with N dimensional vector u such that

$$u^T = (f_1(z), f_2(z), \dots f_N(z)) \tag{4.4}$$

where $f_j(z)$ are 'polynomials' of z. Now one defines M_k which is a $N \times k$ matrix

$$M_k = (u, \partial_- u, \partial_-^2 u, \dots, \partial_-^{k-1} u) \tag{4.5}$$

There are many parameters characterize these polynomials, some of them are moduli parameters of solitons and some of them are redundant. As the moduli parameters change, the configurations can be degenerated. As we have seen in the previous section, solitons cannot collapse and disappear on noncommutative plane. They can still spread all over space, leaving zero energy density. For generic values of the parameters, there would be no common factors among the polynomials $f_j(z)$.

For $k \leq N$, one can define a hermitian projection operator

$$p_k = M_k (\bar{M}_k M_k)^{-1} \bar{M}_k \tag{4.6}$$

which is a uniton with respect to an identity operator. The chiral energy for the group element $h = 2p_k - 1$ would be the masimum degree of the polynomials $f_j(z)$.

We are now ready to adapt the results in ref. [6, 7] on the relation between Toda-model and the uniton method on commutative plane to that on noncommutative plane. Especially we generalize a theorem in ref. [7] to show that a certain class of the solutions of the chiral model on noncommutative plane can be gauge transformed to the solution of the generalized Toda model (4.2) and (4.3) on noncommutative plane. As the following statement is proven, we state it as a theorem.

Theorem 1. *For a N-dimensional vector u in eq. (4.4), we have defined $N \times k$ vector M_k in eq. (4.5) and a projection operator p_k in eq. (4.6). Then the following operator*

$$h = (2p_1 - 1)(2p_2 - 1) \cdots (2p_{N-1} - 1) \tag{4.7}$$

is a solution of the U(N) *chiral model field equation on noncommutative plane. So that $\phi = A_+ = \frac{1}{2} h^{-1} \partial_+ h$ describes the Chern-Simons solitons. Furthermore, there exists a unitary transformation g which leads to $g^\dagger \phi g$ and $g^\dagger A_+ g + g^\dagger \partial_+ g$ are in the Toda ansatz (4.1), and so become the solution of the generalized* U(N) *Toda equations (4.2) and (4.3) on noncommutative plane.*

Proof. The linear algebra with noncommuting variables are full of pit falls. In our case we will see that our scope is just so that some key lore of linear algebra works out fine. The vector u in eq. (4.4) are given by N polynomials of z only. Thus as a function of z we can discuss the linear independence. We assume that the N column vectors $u, \partial_- u, \dots, \partial_-^{N-1} u$ which all depend only on z are linearly independent. (If the degree of $d(u)$ is larger than

N and polynomials are generic, then it is so.) We consider a space $\mathcal{V}$ made of vectors given as a linear combination,

$$V = \sum_{r=1}^{N} (\partial_-^{r-1} u)\, c_r \tag{4.8}$$

where $c_r(z, \bar{z})$'s are scalar functions of z and $\bar{z}$. One start with a unit column vector

$$e_1 = u\, (\bar{u}u)^{-1/2} \tag{4.9}$$

in space $\mathcal{V}$, which shows that $p_1 = e_1 e_1^\dagger$.

As p_2 of eq. (4.6) is a hermitian $N \times N$ projection operators such that $(p_2)^2 = p_2$, we can see that $p_2 u = u, p_2 \partial_- u = \partial_- u$. Now we define another unit column vector

$$e_2 = (1 - p_1)\partial_- u\, \left(\partial_+ \bar{u}(1 - p_1)\partial_- u\right)^{-1/2} \tag{4.10}$$

in $\mathcal{V}$, which is orthogonal to e_1 as $\bar{e}_1 e_2 = 0$. Thus we see that $p_2 e_1 = e_1, p_2 e_2 = e_2$. Now we say $q_2 = e_1 \bar{e}_1 + e_2 \bar{e}_2$, then we see $p_2 q_2 = q_2 = p_2$ on noncommutative space as on commutative space.

By the Gramm-Schmidt process, we choose e_r $(r \geq 2)$ to be a unit vector,

$$e_r = (1 - p_{r-1})\partial_-^{r-1} u \left(\partial_+^{r-1} u (1 - p_{r-1}) \partial_-^{r-1} u \right)^{-\frac{1}{2}} \tag{4.11}$$

which is orthogonal to $e_1, \ldots, e_{r-1}$. Again we see that $p_r = e_1 \bar{e}_1 + \cdots + e_r \bar{e}_r$. Now the unitary matrix defined as

$$g = (e_1, e_2, \ldots, e_N) \tag{4.12}$$

such that $g^{-1} = g^\dagger$ diagonalizes all the p_r's so that

$$g^{-1} p_r g = \mathrm{diag}(1, 1, .., 1, 0, \ldots, 0) \tag{4.13}$$

where $r + 1$ entry is the first zero element. Similar to the case on the commutative plane, one can see that

$$\bar{e}_r \partial_+ e_s \neq 0 \qquad \text{only for } (s = r \quad \text{or} \quad s = r + 1)\,. \tag{4.14}$$

With $g^{-1}(\partial_+ p_r) g = [g^{-1}\partial_+ g, g^{-1} p_r g]$, the above relation implies that

$$[g^{-1}(\partial_+ p_r) g]_{ab} = -\delta_{a,r}\delta_{b,r+1} \bar{e}_r \partial_+ e_{r+1}\,. \tag{4.15}$$

Using the above observation, one can see that the group element h of eq. (4.7) becomes

$$A_+ = \phi = \frac{1}{2} h^{-1} \partial_+ h = \sum_{r=1}^{N-1} \partial_+ p_r\,. \tag{4.16}$$

which is describes the Chern-Simons solitons. Under the gauge transformation by g, the scalar field $g^{-1}\phi g$ and the gauge field $g^{-1} A_+ g + g^{-1}\partial_+ g$ take the form of the Toda-ansatz (4.1),

$$\begin{aligned} h_a &= [g^{-1}\phi g]_{ab} = -\delta_{a,b-1}\, e_a^\dagger \partial_+ e_{a+1}\,, \qquad (a = 1, 2, \ldots, N-1) & (4.17)\\ E_a &= g^{-1} A_+ g + g^{-1}\partial_+ g = \mathrm{diag}(e_1^\dagger \partial_+ e_1, e_2^\dagger \partial_+ e_2, \ldots, e_N^\dagger \partial_+ e_N)\,. & (4.18) \end{aligned}$$

This completes the proof of the above theorem for the Toda-model on noncommutative space.

Let us examine more closely the Toda-model and the Chern-Simons solitons in the U(2) case. The above proof provides a class of the general solution start with any two polynormials $f_1(z)$ and $f_2(z)$ for the u vector (4.4). To feel this general solution, let us examine the simplest ansatz,

$$u = \begin{pmatrix} z \\ c \end{pmatrix} \tag{4.19}$$

with a constant parameter c. (For the rest of discussion, we put $\theta = 2$ and so $z = a$ for simplicity.) The corresponding orthonormal vectors are

$$e_1 = \begin{pmatrix} z \\ c \end{pmatrix} \sqrt{\frac{1}{\mathbb{N} + |c|^2}}, \tag{4.20}$$

$$e_2 = \begin{pmatrix} \bar{c} \\ -\bar{z} \end{pmatrix} \frac{1}{\bar{c}} \sqrt{\frac{|c|^2}{\mathbb{N} + 1 + |c|^2}} \tag{4.21}$$

where $\mathbb{N} = \bar{z}z$ is the number operator on the Hilbert space. With the noncommtative parameter $\theta = 2$ for the simplicity, the scalar field (4.17) becomes

$$h_1 = [g^{-1}\phi g]_{12} = \frac{\sqrt{|c|^2}}{\sqrt{(\mathbb{N} + |c|^2)(\mathbb{N} + 1 + |c|^2)}} \tag{4.22}$$

and the corresponding gauge field (4.18) becomes

$$E_1 = (g^{-1}A_+ g + g^{-1})_{11} = \left(\sqrt{\frac{\mathbb{N} + |c|^2}{\mathbb{N} + 1 + |c|^2}} - 1 \right) a \tag{4.23}$$

$$E_2 = (g^{-1}A_+ g + g^{-1})_{22} = \left(\sqrt{\frac{\mathbb{N} + 2 + |c|^2}{\mathbb{N} + 1 + |c|^2}} - 1 \right) a\,. \tag{4.24}$$

Note that the field strength is diagonal and becomes

$$[g^{-1}(F_{+-})g]_{11} = -[g^{-1}(F_{+-})g]_{22} = \frac{|c|^2}{(\mathbb{N} + |c|^2)(\mathbb{N} + 1 + |c|^2)}\,. \tag{4.25}$$

This is the simplest solution of the U(2) Toda-model or Liouville model on noncommutative plane. Of course we can consider more general solutions, which is quite straightforward.

As noted in ref. [6], one can have a consistent ansatz which is a slightly more general than the ansatz (4.1) for the Toda model. The gauge field in the new ansatz is diagonal as before. The scalar field which had nonzero component only along simple roots of SU(N) has an additional nonzero component, which corresponds to the lowest negative root. This ansatz for the affine Toda model works as well on noncommutative plane. The ansatz for U(N) affine Toda model on noncommutative plane is

$$\begin{aligned} A_+ &= \mathrm{diag}(E_1, E_2, \ldots, E_N)\,, \\ (\phi)_{ab} &= \delta_{a,b-1} h_a \ (a = 1, 2, \ldots, N-1)\,, \qquad \text{except for } (\phi)_{N1} = h_N \end{aligned} \tag{4.26}$$

which now have complex N components E_a and complex N components h_a. The self-dual equations (2.11) and (2.15) on noncommutative plane for this ansatz become

$$\begin{aligned}
\partial_+ h_a + E_a h_a - h_a E_{a+1} &= 0 \qquad (a = 1, 2, \ldots, N-1) \\
\partial_+ h_N + E_N h_N - h_N E_1 &= 0
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
\partial_+(-E_1^\dagger) - \partial_- E_1 + [(-E_1^\dagger), E_1] &= h_1 h_1^\dagger - h_N^\dagger h_N \\
\partial_+(-E_a^\dagger) - \partial_- E_a + [(-E_a^\dagger), E_a] &= -h_{a-1}^\dagger h_{a-1} + h_a h_a^\dagger \qquad (a = 2, \ldots, N-1) \\
\partial_+(-E_N^\dagger) - \partial_- E_N + [(-E_N^\dagger), E_N] &= -h_{N-1}^\dagger h_{N-1} + h_N h_N^\dagger \,.
\end{aligned} \tag{4.28}$$

For $N = 2$, the above model on commutative space with a further restriction on the solution reduces to sinh-Gordon equation. On noncommutative plane, we cannot solve for E_a in terms of h_a and so we cannot get second order equations for h_a only. On commutative plane, the nontrivial solution of the affine Toda-equation has infinite charge Q [6]. It would be very interesting to find some solutions of our generalization of affine Toda-equation on noncommutative plane.

Our generalization of (affine) Toda, Liouville, sinh-Gordon model on noncommutative plane is done directly on the gauge fields and scalar field on noncommutative plane, resulting in coupled first order equations. Thus, our work contrasts to recent works on integrable models on noncommutative plane, where the coupled first order equations are reduced to the coupled second order equations, which is then generalized to noncommutative space. While the later has an appealing feature of having less number of variables, ours has a feature that some of solutions can be obtained explicitly. It remains to been seen whether the two procedures are compatible, and whether the integrability survives in both cases.

5. Moduli dynamics

The moduli space dynamics of chiral field would be

$$\mathcal{K}_{\text{chiral}} = -\frac{1}{2} \int d^2x \, \text{tr}(h^{-1}\dot{h})^2 \tag{5.1}$$

For the simple case $h = 2p - 1$, we get

$$\mathcal{K}_{p^2=p} = 2 \int d^2x \;\; \text{tr}(\dot{p})^2 \tag{5.2}$$

which can be regarded as the soliton dynamics of the purely scalar field on noncommutative plane in large noncommutative limit. Finally, our Chern-Simons theory is first order lagrangian. There are some studies of these moduli space [12, 13, 14, 16]. It provide the metric for the moduli space parameters.

When one consider the moduli dynamics of solitons in the chiral model or the simpler version, CP^N model, some of the parameters of solitons has infinite inertia, or kinetic mess, which implies that those parameters characterizes the vacuum where solitons exist, not the moduli parameters of solitons. The parameters of finite inertia can change with time. The moduli space of solitons on noncommutative space seems complete contrast to that on commutative space. The reason is that the noncommutativity of space provides a short distance cut-off and so solitons cannot collapse and disappear.

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For Chern-Simons solitons the selfdual configuration satisfy a relation (3.3) in a gauge. As they carry zero energy, the lagrangian (2.8) becomes

$$\mathcal{K}_{\rm CS} = \frac{\mathfrak{n} i}{2\pi} \int d^2x \;\; {\rm tr}\, \phi^\dagger \dot{\phi} \tag{5.3}$$

modulo the Gauss law (2.11). Clearly this is a first order in time-derivative and so does not provide any metric on moduli space.

We consider a single uniton solution (3.21) in U(N). We notice that

$$\begin{aligned} \partial_+\partial_- p &= -M\frac{1}{\bar{M}M}\partial_+\bar{M}(1-p)\partial_- M\frac{1}{\bar{M}M}\bar{M} + (1-p)\partial_- M\frac{1}{\bar{M}M}\partial_+\bar{M}(1-p) \\ \partial_t p &= (1-p)\partial_t M\frac{1}{\bar{M}M}\bar{M}\,. \end{aligned} \tag{5.4}$$

The above first order lagrangian (5.3) for the moduli parameters for a single uniton (3.21) becomes

$$\mathcal{K}_{\rm CS} = i\mathfrak{n}\theta\, {\rm Tr}\, {\rm tr}\left[(\partial_+\partial_- p)\partial_t p\right] = 0 \tag{5.5}$$

after integration by part.

The vanishing of the first order lagrangian for the moduli space of a single uniton which represents many fundamental Chern-Simons solitons suggests that the first order lagrangian probably vanishes for all possible self-dual configurations given by Conjecture I. While this remains to be seen, the first order moduli space dynamics between Chern-Simons solitons are expected to be trivial on general ground.

The total angular momentum of many fundamental Chern-Simons solitons described by a single uniton is the sum of that for individual ones and so independent of their moduli parameters. In addition, the fundamental Chern-Simons solitons have integer spin $\mathfrak{n}$ upon quantization. Thus they need no first order interaction in short or long distance for the spin-statistics theorem to hold. This is quite different from the behavior of two anyonic solitons, each with spin s, whose classical angular momentum interpolates between $2s$ in large separation to s^2 in no separation. For anyonic solitons one needs nontrivial first order kinetic terms for the statistical interaction.

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